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The Brouwer fixed point theorem and the transformation rule for multiple integrals via homotopy arguments

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In this note we present an elementary proof of the change of variables formula in multiple integrals, which is based primarily on homotopy arguments. Our crucial homotopy result (see the Lemma) is motivated by the change of variables formula, and is deduced simply by means of differential calculus. As a byproduct of our approach we obtain Brouwer's fixed point theorem, which indicates that the principle homotopy result (Lemma) is intimately connected with the Brouwer degree for mapping.

In the literature we know in fact only one reference [3] related to our approach. In contrast to our note, the aim of [3] is the Brouwer degree of a mapping, and not the change of variables formula. There are other interesting proofs of the change of variables formula; we mention those in [1; 5; 6] in particular.

What does the change of variables formula say? We formulate this as Theorem 1:

Theorem 1. *If ϕ is a diffeomorphism between open subsets Ω and Ω' of $\mathbb{R}^m$, and if f is a function on Ω', then*

$$\int_{\Omega} f \circ \phi \, |\det D\phi| = \int_{\Omega'} f \tag{1}$$

in the sense that if either integral exists, then so does the other and the two are equal.

By definition, a diffeomorphism $\phi\colon \Omega \to \Omega'$ is a bijective mapping which is continuously differentiable in both directions, and $D\phi(x)$ stands for the Jacobian matrix of ϕ at the point x of Ω. The integrals appearing in (1) are always interpreted as Lebesgue integrals. However, if one bears in mind appropriate additional assumptions, they may be understood as (possibly improper) Riemann integrals, too.

In order to prove formula (1) for functions f in $L^1(\Omega')$, i.e. for (equivalence classes of) Lebesgue integrable functions f defined on Ω', we reduce the problem in a standard way (compare [1; (16.22)], [5; 10.8–10.9]).

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Firstly, one need only establish equation (1) for functions in $C_0^0(\Omega')$, that is, for continuous functions vanishing outside a suitable compact subset of Ω'. Indeed, if (1) holds for functions in $C_0^0(\Omega')$, then (1) naturally generates an isometric isomorphism $T(f)=f\circ\phi\,|\det D\phi|$ between the spaces $C_0^0(\Omega')$ and $C_0^0(\Omega)$, provided they are equipped with the norms $\|\cdot\|=\int|\cdot|$. Since $C_0^0(\Omega')$ (resp. $C_0^0(\Omega)$) is dense in $L^1(\Omega')$ (resp. $L^1(\Omega)$), the operator T extends uniquely to an isometry $\tilde{T}$: $L^1(\Omega')\to L^1(\Omega)$ by setting $\tilde{T}(f)=f\circ\phi\,|\det D\phi|$.
Secondly, if $(\Omega'_\alpha)_{\alpha\in\mathscr{A}}$ is any covering of Ω' by open subsets Ω'_α of Ω', it is enough to show in proving Theorem 1 that for all $\alpha\in\mathscr{A}$, $f\in C_0^0(\Omega'_\alpha)$

$$\int f\circ\phi\,|\det D\phi|=\int f \tag{1'}$$

Of course, reduction (1′) is obtained easily via a "partition of unity".

For example one could argue as follows:
Since any $f\in C_0^0(\Omega')$ has compact support $(\operatorname{supp} f=\overline{\{x\,|\,f(x)\neq 0\}})$ in Ω', there exist n open cubes $Q(x_\nu;\delta_\nu)$ (resp. $Q(x_\nu;2\delta_\nu)$) with center x_ν and edge length δ_ν (resp. $2\delta_\nu$) such that each $Q(x_\nu;2\delta_\nu)$ is contained in some Ω'_α, and $\operatorname{supp} f$ lies in $\bigcup_{\nu=1}^n Q(x_\nu;\delta_\nu)$. Choose $0\le\varphi_\nu\in C_0^0(Q(x_\nu;2\delta_\nu))$ satisfying $\varphi_\nu=1$ on $Q(x_\nu;\delta_\nu)$, and define $f_\nu=f\varphi_\nu\dfrac{1}{\sum_{\mu=1}^n\varphi_\mu}$. Now, each f_ν has compact support lying in some Ω'_α, and since $f=\sum_{\nu=1}^n f_\nu$, we derive (1) from equations (1′) by adding.]

Let us sketch the idea used in proving Theorem 1. Suppose that Ω and Ω' are in addition connected, and that f is in $C_0^0(\Omega')$. Denoting by $\varepsilon(\phi)$ the constant sign of $\det D\phi$ in Ω, we may modify (1) to

$$\int f\circ\phi\,\det D\phi=\varepsilon(\phi)\cdot\int f \tag{2}$$

Hence formula (2) tells us that $\int f\circ\phi\,\det D\phi$ does not depend on the diffeomorphism ϕ, provided $\varepsilon(\phi)$ always has the same value. Thus, considering a family $(\phi_\lambda)_{\lambda\in[0,1]}$ of diffeomorphisms ϕ_λ: $\Omega\to\Omega'$ which depend "continuously" on λ, one should be able to show that $\int f\circ\phi_\lambda\det D\phi_\lambda$ is independant of λ for $0\le\lambda\le 1$. This is the basic idea which underlies our approach to the change of variables formula.

Lemma*. *Suppose we are given an open bounded subset Ω of $\mathbb{R}^m$ and a continuous function Φ: $\bar{\Omega}\times[0,1]\to\mathbb{R}^m$, $(x,\lambda)\to\phi_\lambda(x)=\Phi(x,\lambda)$, which is differentiable with respect to x on Ω such that the Jacobian matrix $D\phi_\lambda(x)$ depends continuously on $(x,\lambda)\in\Omega\times[0,1]$.*

Then $\int f\circ\phi_\lambda\det D\phi_\lambda$ does not depend on λ, $0\le\lambda\le 1$, for any continuous function f: $\mathbb{R}^m\to\mathbb{R}$ vanishing in a neighborhood of $\bigcup_{0\le\lambda\le 1}\phi_\lambda(\partial\Omega)$.

* A primary version of this lemma had been used by the second author in his "Antrittsvorlesung" at the University of Munich in 1973.

Here $\partial\Omega$ is the boundary of Ω and $f\circ\phi_\lambda \det D\phi_\lambda$ has compact support in Ω for all $0\leq\lambda\leq 1$.

Before proving the lemma we introduce some notation. If Ω is any open subset of $\mathbb{R}^m$ and k is any nonnegative integer, let $C^k(\Omega)$ be the vector space consisting of all functions f which are k-times continuously differentiable in Ω (Ω may be arbitrary in case $k=0$). The subspace $C_0^k(\Omega)$ consists of all those functions in $C^k(\Omega)$ which have compact support in Ω; clearly we have the embedding $C_0^k(\Omega)\subset C_0^k(\mathbb{R}^m)$. If A and B are nonempty subsets in $\mathbb{R}^m$, then $\operatorname{dist}(A,B)=\inf\{|x-y| \,|\, x\in A,\ y\in B\}$.

Proof of the Lemma. Since Φ is uniformly continuous on $\bar{\Omega}\times[0,1]$, and since f vanishes in a neighborhood of $\bigcup_{0\leq\lambda\leq 1}\Phi_\lambda(\partial\Omega)=\Phi(\partial\Omega\times[0,1])$, there exists a compact subset K of Ω such that $\operatorname{supp} f\circ\phi_\lambda\subset K$ for all $0\leq\lambda\leq 1$. Write $h(\lambda)=\int f\circ\phi_\lambda \det D\phi_\lambda$; we have to show $h'(\lambda)=0$. Let us first assume that $\Phi\in C^2(\Omega\times]0,1[)$, and that $f\in C^1(\mathbb{R}^m)$. Differentiating h, we obtain

$$h'(\lambda)=\int\langle Df\circ\phi_\lambda,\phi'_\lambda\rangle \det D\phi_\lambda+\int f\circ\phi_\lambda \operatorname{trace}(\widetilde{D\phi_\lambda}\cdot D\phi'_\lambda) \tag{3}$$

Here "$'$" denotes the derivative with respect to λ, $\langle\cdot,\cdot\rangle$ means the Euclidean scalar product in $\mathbb{R}^m$, $\widetilde{D\phi_\lambda}$ is the matrix complementary* to the Jacobian matrix $D\phi_\lambda$ and "$\cdot$" stands for the matrix product.

Note that all integrands appearing in (3) have compact support. Thus by partial integration we may remove the derivative $D=(D_1,\ldots,D_m)$ from ϕ'_λ in the second integral of (3). This gives

$$\begin{aligned}\int f\circ\phi_\lambda \operatorname{trace}(\widetilde{D\phi_\lambda}\cdot D\phi'_\lambda)=&-\int\langle Df\circ\phi_\lambda,\phi'_\lambda\rangle\det D\phi_\lambda\\ &-\int f\circ\phi_\lambda\langle\operatorname{div}\widetilde{D\phi_\lambda},\phi'_\lambda\rangle,\end{aligned} \tag{4}$$

where $\operatorname{div}=\langle D,\cdot\rangle$ operates on the columns of $\widetilde{D\phi_\lambda}$. Hence

$$h'(\lambda)=-\int f\circ\phi_\lambda\langle\operatorname{div}\widetilde{D\phi_\lambda},\phi'_\lambda\rangle.$$

But $\operatorname{div}\widetilde{D\phi_\lambda}=0$ (by a perhaps not very well-known identity [2, p. 467]), and therefore $h'(\lambda)=0$ for all $0\leq\lambda\leq 1$.

To prove the lemma in case of general Φ and f, let us choose $I_n=\left]\frac{1}{n},1-\frac{1}{n}\right[$ and an open bounded set Ω_1 satisfying $K\subset\Omega_1\subset\bar{\Omega}_1\subset\Omega$. By mollifying Φ and f, we can find functions Φ_n in $C^2(\Omega_1\times I_n)\cap C^0(\bar{\Omega}_1\times\bar{I}_n)$ and f_n in $C^1(\mathbb{R}^m)$ such that $\Phi_n\to\Phi$ and $D\Phi_n\to D\Phi$ (resp. $f_n\to f$) uniformly on compact subsets of $\Omega_1\times]0,1[$ (resp. $\mathbb{R}^m$) (note $D\Phi(x,\lambda)=D\phi_\lambda(x)$). Moreover, if n and k are nonnegative integers, we may

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* Let A be an $n\times n$ matrix, $A=(a_{ij})$, and let A_{ij} be the $(n-1)\times(n-1)$ matrix obtained from A by deleting the i-th row and j-th column. The matrix $\tilde{A}$ *complementary to* A is defined by letting

$$\tilde{A}=(b_{ij}),\qquad b_{ij}=(-1)^{i+j}\det A_{ji}.$$

assume that f as well as the f_n vanish in a neighborhood of $\Phi_k(\partial\Omega_1 \times I_k)$. By the preceeding step, constants $c_{n,k}$ exist such that $\int f_n \circ (\phi_k)_\lambda \det D(\phi_k)_\lambda = c_{n,k}$ for all λ in I_k. Taking the limits $n \to \infty$ and $k \to \infty$ we get $\int f \circ \phi_\lambda \det D\phi_\lambda = \text{const}$ for all $0<\lambda<1$.

This last equation is valid by continuity also at the endpoints $\lambda=0$ and $\lambda=1$. ■

Using our Lemma and suitable deformations we are now able to prove Theorem 1.

Proof of Theorem 1. *Step 1.* Let us first establish equation (1) for functions f in $C_0^0(\mathbb{R}^m)$ and affine diffeomorphisms $\phi\colon \mathbb{R}^m \to \mathbb{R}^m$, that is $\phi(x) = A \cdot x + b$ with a nonsingular $m \times m$ matrix A. For $0 \le \lambda \le 1$ consider the mapping $\phi_\lambda(x) = \alpha(\lambda) \cdot x + \lambda b$. Here $\alpha(\lambda)$ connects A and $E_A = \operatorname{diag}(\alpha, 1, \dots, 1)$, where $\alpha = \operatorname{sgn}\det A$, continuously within the nonsingular $m \times m$ matrices (see the Appendix). Since $\lim_{|x|\to\infty} |\phi_\lambda(x)| = \infty$ uniformly in $0 \le \lambda \le 1$, every function f in $C_0^0(\mathbb{R}^m)$ vanishes in a neighborhood of $\bigcup_{0 \le \lambda \le 1} \phi_\lambda(\partial B(0;r))$, provided the radius r of the ball $B(0;r)$ centered about 0 is chosen sufficiently large. Thus our Lemma yields $\int f \circ E_A \operatorname{sgn}\det A = \int f \circ \phi \det D\phi$. Since $\int f = \int f \circ E_A$ and $D\phi = A$ we get $\int f = \int f \circ \phi\, |\det D\phi|$.

Step 2. In Step 2 we show that given a diffeomorphism $\phi\colon \Omega \to \Omega'$, we can always find a covering $(\Omega'_\alpha)_{\alpha \in \mathscr{A}}$ of Ω' by open subsets Ω'_α in Ω' such that equations (1′) hold. Choose any z in Ω and consider the diffeomorphism

$$\hat{\phi}(x) = D\phi(z)^{-1} \cdot (\phi(x+z) - \phi(z)). \tag{5}$$

Then a ball $B = B(0;r)$ exists such that $|\hat{\phi}(x) - x| \le \frac{|x|}{2}$ for all x in B. Write $\phi_\lambda = \mathrm{id} - \lambda(\hat{\phi} - \mathrm{id})$; it follows that

$$|\phi_\lambda(x)| \ge |x| - |\hat{\phi}(x) - x| \ge \frac{|x|}{2} \tag{6}$$

for every x in B and every $0 \le \lambda \le 1$. Put $B' = B(0; r/2)$; then in view of (6) every function f in $C_0^0(B')$ vanishes in a neighborhood of $\bigcup_{0 \le \lambda \le 1} \phi_\lambda(\partial B)$. Hence our Lemma yields

$$\int f = \int f \circ \hat{\phi} \det D\hat{\phi} = \int f \circ \hat{\phi}\, |\det D\hat{\phi}|,$$

the last equality being valid since $\det D\hat{\phi}(0) = 1$.

Finally, it is an easy task to verify that if Theorem 1 (admitting only functions having compact support) is true for diffeomorphisms ϕ and ψ, then Theorem 1 holds for the composed diffeomorphism $\phi \circ \psi$, too. Thus, in view of (5) and Step 1, there exists a neighborhood Ω_z of z in Ω such that ϕ satisfies (1) for every function f in $C_0^0(\phi(\Omega_z))$. Hence we have proven (1′) with respect to the covering $(\phi(\Omega_z))_{z \in \Omega}$ of Ω' and have thereby proven Theorem 1. ■

Though our Lemma served primarily to give an elementary proof of the change of variables formula, it may be further exploited to give a short demonstration of the Brouwer Fixed Point Theorem.

Theorem 2. *Every continuous mapping ϕ from the closed unit ball to itself possesses at least one fixed point.*

Proof. If $\phi(x)\neq x$ for all x in the closed unit ball B, then the deformation $\phi_\lambda = \mathrm{id}-\lambda\phi$ would satisfy the estimates

$$|\phi_1(x)|=|\phi(x)-x|>0 \qquad \text{for } x \text{ in } B$$
$$|\phi_\lambda(x)|\geq|x|-\lambda|\phi(x)|\geq 1-\lambda>0 \qquad \text{for } x\in\partial B \text{ and } 0\leq\lambda<1.$$

Since a continuous function attains its infimum on each compact set, a positive constant δ would exist such that

$$|\phi(x)-x|>\delta \qquad \text{for every } x\in B$$
$$|\phi_\lambda(x)|>\delta \qquad \text{for every } x\in\partial B \text{ and all } 0\leq\lambda\leq 1. \tag{7}$$

Without loss of generality we may assume ϕ to be continuously differentiable on $B(0;1)$ (otherwise approximate ϕ by a smooth function so that (7) is satisfied with δ replaced by $\delta/2$). Clearly, every function f in $C_0^0(B(0;\delta))$ vanishes in a neighborhood of $\bigcup_{0\leq\lambda\leq 1}\phi_\lambda(\partial B)$, thus our Lemma yields $\int f=\int f\circ\phi_1 \det D\phi_1$. In view of (7) $f\circ\phi_1$ vanishes everywhere, which shows that $\int f=0$. Since functions $f\in C_0^0(B(0;\delta))$ with $\int f\neq 0$ do indeed exist, we arrive at a contradiction. ∎

The proof of Theorem 2 presented above suggests that there is more to our Lemma than meets the eye. Indeed, let Ω be any bounded open subset of $\mathbb{R}^m$, and let $\phi\colon \bar{\Omega}\to\mathbb{R}^m$ be continuously differentiable in Ω. If y is a point in $\mathbb{R}^m$ satisfying $\rho=\mathrm{dist}(\{y\},\phi(\partial\Omega))>0$, then for any f in $C_0^0(B(y;\rho))$ with $\int f=1$, the degree of ϕ with respect to Ω and y may be defined as

$$d(\phi,\Omega,y)=\int f\circ\phi \det D\phi. \tag{8}$$

Suppressing for the moment the (apparent) dependance of $d(\phi,\Omega,y)$ on f, our Lemma shows precisely that $d(\phi,\Omega,y)$ is homotopy-invariant. Indeed, if $\Phi\colon \bar{\Omega}\times[0,1]\to\mathbb{R}^m$, $(x,\lambda)\to\phi_\lambda(x)$ is any parameter family of maps satisfying the assumptions of the Lemma, and if $y\notin\phi_\lambda(\partial\Omega)$ for each λ, then $d(\phi_\lambda,\Omega,y)$ does not depend on $0\leq\lambda\leq 1$. It is worth mentioning that the most important property of the Brouwer degree, namely the homotopy invariance, is an immediate consequence of our approach to the change of variables formula.

Let us now see why the right hand side of (8) really does not depend on the choice of f. For simplicity we assume $y=0$ and $\phi\in C^2(\Omega)$. Consider functions f and g in $C_0^0(B(0;\rho))$ satisfying $\int f=1=\int g$. Write $\mu=f-g$ and take any $j\in C_0^1(B(0;\varepsilon))$, where

$$0<\varepsilon<\mathrm{dist}(\operatorname{supp}\mu,\partial B(0;\rho)).$$

Define $v(z)=\int \int_1^0 j(z-t\,x)\,x\,\mu(x)\,dt\,dx$. Then v is continuously differentiable, has compact support in $B(0;\rho)$ and

$$\begin{aligned}
\operatorname{div} v(z) &= \int \left(\int_1^0 \langle D\,j(z-t\,x), x\rangle\, dt\right) \mu(x)\,dx \\
&= \int \left(\int_0^1 \frac{d}{dt}(j(z-t\,x))\,dt\right) \mu(x)\,dx \\
&= \int j(z-x)\,\mu(x)\,dx - j(z)\cdot \int \mu(x)\,dx \\
&= \int j(z-x)\,\mu(x)\,dx, \text{ since } \int \mu(x)\,dx = 0
\end{aligned}$$

Thus with the abbreviation $(j*\mu)(z)=\int j(z-x)\,\mu(x)\,dx$ we have $\operatorname{div} v = j*\mu$. Write $w = \widetilde{D\phi}\cdot(v\circ\phi)$. Then

$$\operatorname{div} w = \operatorname{trace}\,(D\,\phi\cdot\widetilde{D\phi}\cdot(D\,v\circ\phi)) + \langle \operatorname{div}\widetilde{D\phi}, v\circ\phi\rangle.$$

Recall that $D\,\phi\cdot\widetilde{D\phi}=(\det D\,\phi)\,E$ and $\operatorname{div}\widetilde{D\phi}=0$, so that

$$\operatorname{div} w = (\operatorname{div} v)\circ\phi \det D\,\phi = (j*\mu)\circ\phi \det D\,\phi.$$

Hence $0=\int \operatorname{div} w = \int (j*\mu)\circ\phi \det D\,\phi$ and thus

$$\int (j*f)\circ\phi \det D\,\phi = \int (j*g)\circ\phi \det D\,\phi. \tag{9}$$

If we put in (9) a sequence (j_n) of functions $j_n \in C_0^1\left(B\left(0;\frac{1}{n}\right)\right)$ satisfying $\int j_n = 1$, we conclude that $\int f\circ\phi \det D\,\phi = \int g\circ\phi \det D\,\phi$ since $j_n * f \to f$ and $j_n * g \to g$ uniformly on $\mathbb{R}^m$.

Of course, starting with formula (8) one may develop a complete treatment of degree theory. We have not done so here, since the primary aim of this note is the change of variables formula. We mention the articles [3; 4] in which such treatments are presented.

Appendix

Denote by $GL(m,\mathbb{R})$ the set of nonsingular realvalued $m\times m$ matrices.

If $A\in GL(m,\mathbb{R})$, then a continuous mapping α: $[0,1]\to GL(m,\mathbb{R})$ exists satisfying $\alpha(0)=A$ and $\alpha(1)=\operatorname{diag}(\sigma,1,\dots,1)$, where $\sigma=\operatorname{sgn}\det A$.

Proof. By a well known theorem of linear algebra we have a representation of the form $D=B_1\cdot\ldots\cdot B_k\cdot A$, where D is a diagonal matrix containing only $+1$ or -1 on the diagonal. Furthermore, each B_κ is a matrix of type $E_j^i(c)$ having the number c in the i-th row and the j-th column and being identical with the unit matrix E otherwise. In addition we have $c>0$ if $i\neq j$. Thus using the mappings $[0,1]\ni\lambda\to E_j^i(1+\lambda(c-1))$ as well as $[0,1]\ni\lambda\to E_j^i(\lambda\,c)$ if $i\neq j$, the matrix A may be deformed continuously into D within the class $GL(m,\mathbb{R})$. Now, fix any pair of numbers -1 in the diagonal of D and modify D to

$$\beta(\lambda)=\begin{pmatrix} \ddots & & & & \\ & \cos\lambda\pi & \dots & -\sin\lambda\pi & \\ & \vdots & \ddots & \vdots & \\ & \sin\lambda\pi & \dots & \cos\lambda\pi & \\ & & & & \ddots \end{pmatrix}$$

for any $0\leq\lambda\leq 1$. By a finite number of such deformations β: $[0,1)\to GL(m,\mathbb{R})$ we may deform D continuously into the matrix $\operatorname{diag}(\sigma,1,\dots,1)$ with $\sigma=\operatorname{sgn}\det A$. ■

Notes added in proof. 1. S.D. Chatterji has kindly pointed out to us the following closely related papers (all published in the American Mathematical Monthly): J. Milnor, Analytic proofs of the "Hairy Ball Theorem" and the Brouwer fixed-point theorem, *85*, 521–524 (1978); Y. Kannai, An elementary proof of the no-retraction theorem, *88*, 264–268 (1981); H.W. Siegberg, Some historical remarks concerning degree theory, *88*, 125–139 (1981).

2. There is a very natural proof of the identity $\operatorname{div}\widetilde{D\phi}=0$ in C.B. Morrey, Jr., Multiple integral problems in the calculus of variations and related topics, University of California Press, Berkeley and Los Angeles, 1943; chapter II, Lemma 1.1. The proof there uses simply induction on the space dimension m.

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