Math. Ann. 242, 85–96 (1979)

Mathematische Annalen

A Geometric Proof of the Spectral Theorem for Unbounded Self-Adjoint Operators

Herbert Leinfelder

Universität Bayreuth, Fachbereich Mathematik und Physik, Postfach 3008, D-8580 Bayreuth, Federal Republic of Germany

In this note we are concerned with unbounded self-adjoint operators in a Hilbert space. Denoting such an Operator by A we give a direct and geometric demonstration of the facts associated with the formula

$$A = \int_{-\infty}^{\infty} \lambda dE(\lambda) .$$

That is, we establish directly the well-known spectral theorem for unbounded self-adjoint operators using only simple geometric intrinsic properties of Hilbert space.

Of course the fundamental facts about the spectral representation of bounded as well as unbounded operators have been known in substance since the appearance in 1906 [3] of Hilbert's memoir on integral equations and in 1929 [8] of von Neumann's fundamental paper on unbounded operators. Since that time many papers have been published in this subject using a variety of methods. Some of these methods apply only to bounded operators, while others are suited to the general (unbounded) case. However, all but a few of these methods use techniques and principles which lie outside of Hilbert space theory proper, such as Helly's selection principle, Riesz's second representation theorem and so on. For further examples of technique and for a comprehensive list of references on the spectral theorem we refer the reader to [1], p. 927 and [5].

It was not until 1935 that Lengyel and Stone [5] gave a new proof of the spectral theorem which was strictly elementary in the sense that it depended only upon intrinsic properties of Hilbert space. Their paper dealt with the case of bounded operators and the authors remarked that they could not handle the general case in the same way. In their introduction they wrote: "Indeed our method, which requires the study of powers of an operator, is not suited to the case of unbounded operators" ([5], p. 853). Just this sentence stimulated the author to try to handle the general case in the same way and indeed it is possible.

The fundamental idea of our proof of the spectral theorem as well as that of Lengyel and Stone consists in considering invariant subspaces

$$F(A, \lambda) = \{x \in H | x \in D(A^n), \|A^n x\| \leqq \lambda^n \cdot \|x\| \ \text{ for } \ n = 1, 2, 3, \dots\} .$$

0025-5831/79/0242/0085/$02.40

But now, in the unbounded case, one main difficulty arises, namely to prove the density of $\bigcup_{\lambda \geqq 0} F(A, \lambda)$, whereas this is trivial for bounded operators A. This difficulty is overcome in the fundamental Lemma 4. On the other hand Lemma 4 itself is critically dependent on Lemma 3 which enables us to extend well-known properties (Lemma 2) of finite dimensional symmetric transformations to unbounded self-adjoint operators by a simple approximation process (compare corollary). For Theorem 1 we give the announced geometric proof of the spectral theorem. In case A is semibounded, for instance $A \geqq 0$, the spectral family of A can be defined immediately by the formula

$$E(\lambda)=\begin{cases}\mathrm{Proj}_{F(A,\lambda)} & \text{if } \lambda \geqq 0 \\ 0 & \lambda<0,\end{cases}$$

whereas in the non-semibounded case a simple limiting process is needed [for details see formula (18)]. In addition we want to remark that the uniqueness proof of the spectral family given here can be carried over literally to the case of unbounded normal operators and then yields a short proof of a famous theorem of Fuglede (compare the concluding remarks, [2], pp. 66–69 and [9], Theorem 1.16). Finally in Theorem II we show, by analysing our proof of the spectral theorem, that any closed symmetric operator possesses a unique maximal self-adjoint part.

In the following the (usual) notations and definitions are used in the same way as in the book "Perturbation Theory for Linear Operators" by Kato [4].

For closed linear operators A with domain $D(A)$ in the complex Hilbert space $(\mathrm{H}, \langle \cdot, \cdot \rangle)$ let us consider

$$D^{\infty}(A):=\bigcap_{n=1}^{\infty} D(A^n)$$

and for any real number $\lambda \geqq 0$ the set

$$F(A, \lambda):=\{x \in D^{\infty}(A) \mid \|A^n x\| \leqq \lambda^n \cdot \|x\| \ \text{ for } n=1,2,3,\ldots\}. \tag{1}$$

In general $F(A, \lambda)$ is not a closed linear subspace of H, but in the case where A is a symmetric closed operator, we can prove this.

Lemma 1. *Suppose A is a symmetric closed linear operator in the Hilbert space H and λ is a non-negative real number. Then*

i) *$F(A, \lambda)$ is a closed linear subspace of H, which is left invariant by the operator A.*

ii) *Every bounded linear operator B satisfying $B \cdot A \subset A \cdot B$ maps $F(A, \lambda)$ into itself; similarily we have $B^*(F(A, \lambda)^{\perp} \subset F(A, \lambda)^{\perp}$.*

Remarks. 1) It should be noticed that Lemma 1 remains valid even for normal operators A, as is seen immediately by inspecting the proof of Lemma 1.

2) In case A is bounded we refer to [2], p. 66, for Lemma 1.

Proof. First the case $\lambda=0$ is trivial, because $F(A, 0)$ is equal to the null space of A, which obviously is a closed linear subspace. Now suppose $\lambda>0$.

Similarily to $F(A,\lambda)$ we consider the space

$$G(A,\lambda) := \{x \in D^\infty(A) \mid \text{There exist a } c = c(x) > 0 \text{ such that } \|A^n x\| \leqq \lambda^n c \text{ for } n = 1,2,3,\ldots\} . \tag{2}$$

Clearly $G(A,\lambda)$ is a linear subspace of H and $F(A,\lambda) \subset G(A,\lambda)$. Suppose there is an $x \in G(A,\lambda)$ belonging not to $F(A,\lambda)$ with $\|x\| = 1$. Then there exists an integer m such that $\|A^m x\| > \lambda^m$. With $S := \frac{1}{\lambda^m} A^m$ we have $\|Sx\| > 1$. Now for every integer l the following inequalities hold

$$\|S^{2^l} x\|^2 = \langle S^{2^l} x, S^{2^l} x\rangle = \langle x, S^{2^{l+1}} x\rangle \leqq \|S^{2^{l+1}} x\|$$

hence $\|S^{2^l} x\| \geqq \|Sx\|^{2^l}$ and consequently the sequence $(\|S^{2^l} x\|)$ tends to infinity. This contradicts the assumption $x \in G(A,\lambda)$, so we must have $F(A,\lambda) = G(A,\lambda)$, which shows that $F(A,\lambda)$ is a linear subspace.

In order to prove the closedness of $F(A,\lambda)$ we consider a sequence $(x_n) \subset F(A,\lambda)$, $x_n \to x$. Then $x_n - x_m \in F(A,\lambda)$ and for every integer l we have

$$\|A^l x_n - A^l x_m\| = \|A^l(x_n - x_m)\| \leqq \lambda^l \cdot \|x_n - x_m\| .$$

Hence there is a $y_l \in H$ such that $A^l x_n \to y_l$. The closedness of A implies by induction that $x \in D(A^l)$ and $y_l = A^l x$ for each integer l. This means $x \in D^\infty(A)$. The inequalities $\|A^l x_n\| \leqq \lambda^l \|x_n\|$ yield $\|A^l x\| \leqq \lambda^l \|x\|$ as $n \to \infty$. So we have proved $x \in F(A,\lambda)$. To show $A(F(A,\lambda)) \subset F(A,\lambda)$ we take $x \in F(A,\lambda)$, then $\|A^n(Ax)\| = \|A^{n+1}x\| \leqq \lambda^{n+1}\|x\| = \lambda^n \cdot (\lambda \cdot \|x\|)$ which yields $Ax \in G(A,\lambda) = F(A,\lambda)$.

Suppose now B is a bounded linear operator satisfying $B \cdot A \subset A \cdot B$ and let be $x \in F(A,\lambda)$. By induction we conclude $Bx \in D^\infty(A)$ and $A^n Bx = BA^n x$ for every integer n. Then $\|A^n(Bx)\| = \|B(A^n x)\| \leqq \|B\| \cdot \|A^n x\| \leqq \|B\| \cdot \lambda^n \cdot \|x\|$ and $Bx \in F(A,\lambda)$ is proved. The inclusion $B^*(F(A,\lambda)^\perp) \subset F(A,\lambda)^\perp$ now is a trivial consequence. □

In the case of finite dimensional Hilbert spaces H we have the following description of $F(A,\lambda)^\perp$.

Lemma 2. *Suppose A is a symmetric linear operator in a finite dimensional Hilbert space H. Then we have for all $x \in F(A,\lambda)^\perp$, $x \neq 0$, and all $\lambda \geqq 0$.*

i) $\|Ax\| > \lambda\|x\|$,

ii) $\langle Ax, x\rangle > \lambda\|x\|^2$ *if* $A \geqq 0$.

Proof. Ad ii). Define $\mu := \min\{\langle Ax, x\rangle \mid \|x\| = 1, x \in F(A,\lambda)^\perp\} \geqq 0$. It is clear that μ is an eigenvalue of $A|F(A,\lambda)^\perp$, hence $F(A,\mu) \cap F(A,\lambda)^\perp \neq 0$. Because of $F(A,\mu) \subset F(A,\lambda)$ if $\mu \leqq \lambda$, we must have $\mu > \lambda$, which shows ii).

Ad i). We proceed in a similar way and define $\mu := \min\{\|Ax\| \mid \|x\| = 1, x \in F(A,\lambda)^\perp\}$. Due to $\mu^2 = \min\{\langle A^2 x, x\rangle \mid \|x\| = 1, x \in F(A,\lambda)^\perp\}$ we know that μ^2 is an eigenvalue of B^2, where $B := A|F(A,\lambda)^\perp$. The equality

$$0 = \det(B^2 - \mu^2) = \det(B - \mu) \cdot \det(B + \mu)$$

tells us that μ or $-\mu$ is an eigenvalue of B. From this we conclude $F(A,\mu) \cap F(A,\lambda)^\perp \neq 0$. Similarly as above we have $\mu > \lambda$, hence the statement. □

The following approximation-lemma is one key to our proof of the spectral theorem for unbounded operators, because it enables us to extend the decisive inequalities of Lemma 2 to the case of unbounded symmetric operators.

Lemma 3. *Suppose A is a closed symmetric operator in a Hilbert space H and λ, μ are non-negative real numbers. Then for every $x \in F(A,\mu) \cap F(A,\lambda)^{\perp}$ there is a sequence (A_n) of symmetric operators $A_n: H_n \to H_n$ defined in the finite dimensional spaces $H_n \subset H$ and a sequence (x_n) of Elements x_n belonging to H_n in such a way that*

$$\begin{cases} x_n \in F(A_n, \lambda)^{\perp} \\ \lim\limits_{n\to\infty} \|x_n - x\| = 0 = \lim\limits_{n\to\infty} \|A_n x_n - Ax\| \,. \end{cases} \tag{3}$$

If $A \geqq 0$, then A_n can be chosen non-negative.

Remark. In principle the idea of Lemma 3 goes back to Lengyel and Stone [5].

Proof. We take $x \in F(A,\mu) \cap F(A,\lambda)^{\perp}$ and fix it for the rest of the proof. Let us define

$$H_n := \text{linear hull } \{x, Ax, \ldots, A^n x\} \quad \text{for} \quad n = 1, 2, \ldots,$$

$$H_\infty := \text{closed linear hull } \{A^n x \mid n = 0, 1, 2, \ldots\} \,.$$

A short calculation shows $A(H_\infty) \subset H_\infty \subset F(A,\mu) \cap F(A,\lambda)^{\perp}$ and $x, Ax \in H_n \subset H_{n+1}$; $\dim H_n \leqq n+1$; $\overline{\bigcup\limits_{n=1}^{\infty} H_n} = H_\infty$.

Now we introduce the operators

$$P_n := \mathrm{Proj}_{H_n}, \quad P := \mathrm{Proj}_{H_\infty}, \quad \hat{A}_n := P_n A P_n, \quad A_n := \hat{A}_n | H_n \,.$$

We note that A_n, $\hat{A}_n$ are symmetric and bounded. In case $A \geqq 0$ we have $A_n \geqq 0$ too.

Because of $P_n \xrightarrow{s} P$ and $\|A|H_\infty\| \leqq \mu$, it is easy to see that $\hat{A}_n \xrightarrow{s} PAP$ and more generally $\hat{A}_n^l \xrightarrow{s} (PAP)^l$ for all integers l. This yields the relation

$$\lim_{n\to\infty} \|\hat{A}_n^l y - A^l y\| = 0 \qquad (y \in H_\infty); \tag{4}$$

according to Riesz's decomposition theorem (relative to H_n) we get sequences $(x_n) \subset F(A_n, \lambda)^{\perp}$, $(y_n) \subset F(A_n, \lambda)$ such that

$$x = x_n + y_n \,.$$

Since $\|x\|^2 = \|x_n\|^2 + \|y_n\|^2$, the sequence (y_n) is bounded. Hence there exists an y such that without loss of generality

$$y_n \xrightarrow{w} y \in H_\infty \subset F(A,\lambda)^{\perp} \,. \tag{5}$$

Our aim is to show $y = 0$ and for doing that it is sufficient to prove $y \in F(A,\lambda)$. For all integers l we have

$$\begin{aligned} \|A^l y\|^2 = \langle A^l y, A^l y\rangle = \langle y, A^{2\cdot l} y\rangle \underset{(4),(5)}{=} &\lim_{n\to\infty} \langle y_n, \hat{A}_n^{2\cdot l} y\rangle \\ = \lim_{n\to\infty} \langle \hat{A}_n^{2\cdot l} y_n, y\rangle \leqq &\lim_{n\to\infty} \sup \|A_n^{2\cdot l} y_n\| \cdot \|y\| \\ = \lim_{n\to\infty} \sup \lambda^{2\cdot l} &\|y_n\| \cdot \|y\| \leqq \lambda^{2\cdot l} \|x\| \cdot \|y\| \,. \end{aligned}$$

Hence $y \in G(A,\lambda) = F(A,\lambda)$, which yields $y = 0$.

Because of $\|x-x_n\|^2=\|y_n\|^2=\langle x,y_n\rangle\to\langle x,0\rangle=0$

$$Ax=P_nAP_nx$$

and

$$\begin{aligned}\|A_nx_n-Ax\|&=\|P_nAP_nx_n-P_nAP_nx\|\\&\leqq\|P_nAP_n\|\cdot\|x_n-x\|\\&\leqq\mu\cdot\|x_n-x\|\,.\end{aligned}$$

We finally arrive at $\lim_{n\to\infty}\|x_n-x\|=0=\lim_{n\to\infty}\|A_nx_n-Ax\|$. □

Corollary. *Let A be a closed symmetric operator in H, and $\lambda,\mu>0$ real numbers. Then for all $x\in F(A,\mu)\cap F(A,\lambda)^{\perp}$*

i) $\|Ax\|\geqq\lambda\|x\|$,

ii) $\langle Ax,x\rangle\geqq\lambda\|x\|^2$ *if* $A\geqq0$.

Proof. This is a simple consequence of Lemmas 2 and 3 and a limiting process. □

The following lemma is the other decisive step to our proof of the spectral theorem.

Lemma 4. *Let A be a closed symmetric operator in H. Then the following statements are equivalent:*

i) $\bigcup_{n=1}^{\infty}F(A,n)$ *is dense in H,*

ii) *A is self-adhoint.*

Remark. Lemma 4 is closely related to Nelson's theorem [7] on analytic vectors (see also [6]).

Proof. Suppose $\bigcup_{n=1}^{\infty}F(A,n)$ is dense in H. Clearly $A\subset A^*$, so we have to show $D(A^*)\subset D(A)$. Define $P_n:=\mathrm{Proj}_{F(A,n)}$ and take an $x\in D(A^*)$, then $P_nx\xrightarrow{s}x$. Because of

$$A^*x=A^*(x-P_nx)+AP_nx$$

$$\langle A^*(x-P_nx),AP_nx\rangle=\langle x-P_nx,A^2P_nx\rangle=0$$

we have $\|AP_nx\|\leqq\|A^*x\|$. Hence without loss of generality (AP_nx) is weakly convergent. As an immediate consequence of the weak-closedness of A we have $x\in D(A)$.

Now, suppose A is self-adjoint. We take a $\lambda\in\mathbb{C}\setminus\mathbb{R}$ and define with $R(A,s):=(A-s)^{-1}$ the symmetric bounded operator

$$B:=\frac{R(A,\lambda)-R(A,\lambda^*)}{\lambda-\lambda^*}=R(A,\lambda^*)\cdot R(A,\lambda)$$

(here we make use of the fact that H is a complex Hilbert space). We wish to prove for all $\varepsilon>0$

$$F(B,\varepsilon)^{\perp}\subset F\left(A,|\lambda|+\frac{\|R(A,\lambda)\|}{\varepsilon}\right). \tag{6}$$

For $\varepsilon>0$ we consider the bounded operator

$$B_\varepsilon : F(B,\varepsilon)^\perp \to F(B,\varepsilon)^\perp , \qquad B_\varepsilon x := Bx .$$

Notice that $F(B,\varepsilon)^\perp$ is an invariant subspace relative to B. According to our corollary (with $\lambda:=\varepsilon$, $\mu:=\|B\|$) we get

$$\|B_\varepsilon x\| \geqq \varepsilon\|x\| \quad \text{for all} \quad x\in F(B,\varepsilon)^\perp . \tag{7}$$

Because of (7) it is clear, that B_ε is one-to-one and has closed range $R(B_\varepsilon)$. Since B_ε is symmetric, $R(B_\varepsilon)=N(B_\varepsilon)^\perp=F(B,\varepsilon)^\perp$; hence B_ε is bijective. Now for every $x\in F(B,\varepsilon)^\perp$ there exists an $x'\in F(B,\varepsilon)^\perp$ with

$$x=B_\varepsilon x'=Bx'=R(A,\lambda^*)\cdot R(A,\lambda)x'\in D(A) , \tag{8}$$

$$Ax=\lambda^* x+R(A,\lambda)x' . \tag{9}$$

Since B commutes with $R(A,\lambda)^*=R(A,\lambda^*)$ we conclude $R(A,\lambda)x'\in F(B,\varepsilon)^\perp$ according to Lemma 1 ii), hence $Ax\in F(B,\varepsilon)^\perp$. Iterating the preceding method, we get $A^n x\in F(B,\varepsilon)^\perp \subset D(A)$, which means $x\in D^\infty(A)$.

Using (7)–(9) we estimate $\|Ax\|$ by

$$\begin{aligned}\|Ax\| &\leqq |\lambda|\cdot\|x\|+\|R(A,\lambda)\|\cdot\|x'\|\\ &\leqq |\lambda|\cdot\|x\|+\|R(A,\lambda)\|\cdot\frac{\|B_\varepsilon x'\|}{\varepsilon}\\ &\leqq |\lambda|\cdot\|x\|+\frac{\|R(A,\lambda)\|}{\varepsilon}\cdot\|x\| .\end{aligned}$$

Iterating this inequality we obtain $x\in F\left(A, |\lambda|+\dfrac{\|R(A,\lambda)\|}{\varepsilon}\right)$ and we have shown formula (6).

The rest of the proof now is a simple formal manipulation. For integers $n>|\lambda|$ we put $\varepsilon_n:=\dfrac{\|R(A,\lambda)\|}{n-|\lambda|}$ and (6) becomes

$$F(B,\varepsilon_n)^\perp \subset F(A,n)$$

which gives us

$$\bigcap_{n>|\lambda|} F(A,n)^\perp \subset \bigcap_{n>|\lambda|} F(B,\varepsilon_n) . \tag{10}$$

Since $\varepsilon_n\to 0$, we have $\bigcap_{n>|\lambda|} F(B,\varepsilon_n)=0$, for B is one-to-one. Now (10) shows us, that $\bigcup_{n=1}^{\infty} F(A,n)$ is dense in H. □

Remarks. 1) In the case the operator A is semibounded, the proof of Lemma 4 simplifies considerably. Suppose $A\geqq 1$ (without loss of generality), then (6) can be replaced by the handier formula

$$F(A^{-1},\varepsilon^{-1})^\perp \subset F(A,\varepsilon) . \tag{11}$$

2) If A is a normal operator, assertion i) of Lemma 4 remains valid as is easily seen by the inclusion $F(A^*A,\varepsilon^2)\subset F(A,\varepsilon)$ and the self-adjointness of A^*A.

Probably the next lemma is well-known. We state it here, because we didn't find a suitable reference.

Lemma 5. *Let A be a symmetric closed linear operator in H and (P_n) an increasing sequence of projections such that $R(P_n) \subset D(A)$, $AP_n = P_n A P_n$, and $P_n \xrightarrow{s} \mathrm{Id}$.*
Then the following assertions hold:

$$\begin{cases} D(A) = \{x \in H \mid (AP_n x) \text{ converges}\} = \{x \in H \mid (\|AP_n x\|) \text{ converges}\} \\ Ax = s\text{-}\lim\limits_{n \to \infty} AP_n x \text{ for all } x \in D(A)\,. \end{cases} \tag{12}$$

Proof. Lemma 5 is an immediate consequence of the identities

$$\|AP_l x - AP_n x\|^2 = \|AP_l x\|^2 - \|AP_n x\|^2 \qquad (x \in H,\ l \geqq n)\,, \tag{13}$$

$$\|Ax - AP_n x\|^2 = \|Ax\|^2 - \|AP_n x\|^2 \qquad (x \in D(A)) \tag{14}$$

which follow easily from the Pythagorean theorem.

Let x be in $D(A)$. From (14) we conclude $\|AP_n x\| \leqq \|Ax\|$ and (13) shows that the sequence $(\|AP_n x\|)$ is increasing, hence convergent. Equation (13) now tells us that $(AP_n x)$ is a Cauchy-sequence, so it converges to Ax due to the closedness of A. The other inclusions are obvious. □

After these preparatory lemmas we are able to give our direct proof of the spectral theorem for unbounded self-adjoint operators.

Theorem I. *Every self-adjoint operator A in the Hilbert space H admits one and only one spectral family $(E(\lambda))_{\lambda \in \mathbb{R}}$ such that*

$$D(A) = \left\{ x \in H \,\middle|\, \int_{\mathbb{R}} \lambda^2 d\langle E(\lambda)x, x\rangle < \infty \right\}$$

$$\langle Ax, y\rangle = \int_{\mathbb{R}} \lambda d\langle E(\lambda)x, y\rangle \qquad (x \in D(A),\ y \in H)\,.$$

Proof. Uniqueness: Suppose $(E(\lambda))_{\lambda \in \mathbb{R}}$ is a spectral family of A. First we prove for all $\lambda \geqq 0$

$$F(A, \lambda) = R(E[-\lambda, \lambda])\,. \tag{15}$$

Let x be in $R(E[-\lambda, \lambda])$, then $x = E[-\lambda, \lambda]x \in D^\infty(A)$ and

$$\|A^n x\|^2 = \int_{[-\lambda,\lambda]} \mu^{2\cdot n} d\langle E(\mu)x, x\rangle \leqq \lambda^{2\cdot n} \int_{[-\lambda,\lambda]} d\langle E(\mu)x, x\rangle = \lambda^{2\cdot n}\|x\|$$

so $x \in F(A, \lambda)$. Now suppose x is an element of $F(A, \lambda)$. As a consequence of Lemma 1i) and the preceding step, we have $x - E[-\lambda, \lambda]x \in F(A, \lambda)$. We put $y = x - E[-\lambda, \lambda]x$ and get

$$O \geqq \|Ay\|^2 - \lambda^2\|y\|^2 = \int_{\mathbb{R}\setminus[-\lambda,\lambda]} (\mu^2 - \lambda^2) d\langle E(\mu)y, y\rangle\,.$$

This shows $O = \langle E(\mathbb{R} \setminus [-\lambda, \lambda])y, y\rangle = \|y\|^2$, so $x = E[-\lambda, \lambda]x$ and $x \in R(E[-\lambda, \lambda])$ is proved.

In order to complete the proof of uniqueness we only have to state that for all $\lambda, \mu \in \mathbb{R}$, $\lambda < \mu$ the following equality holds

$$R(E[\lambda, \mu]) = F\left(A - \frac{\lambda+\mu}{2}, \frac{\mu-\lambda}{2}\right). \tag{16}$$

Existence: For any integer n let be $P_n := \mathrm{Proj}_{F(A,n)}$ and $A_n := A|F(A,n)$. Then A_n is a symmetric operator with $\|A_n\| \leqq n$, and

$$R(P_n) \subset D(A) \quad AP_n = P_n A P_n, \quad P_n \xrightarrow{s} \mathrm{Id} \tag{17}$$

holds due to Lemmas 1 and 4. We define

$$E_n(\lambda) := \begin{cases} \mathrm{Proj}_{F(A_n+n, \lambda+n)} & \text{if} \quad \lambda \geqq -n \\ 0 & \text{if} \quad \lambda < -n \end{cases} \tag{18}$$

and obtain as an easy consequence

$$\begin{aligned} &E_n(n) = P_n, \quad E_n(-n-\varepsilon) = 0 \quad \text{for each} \quad \varepsilon > 0 \\ &E_n(\lambda) \leqq E_n(\mu) \quad \text{if} \quad \lambda \leqq \mu \\ &E_n(\lambda) = s\text{-}\lim_{\substack{\mu \to \lambda \\ \mu > \lambda}} E_n(\mu). \end{aligned} \tag{19}$$

Next we establish the fundamental relations

$$\langle A_n x, y\rangle = \int_{\mathbb{R}} \lambda d\langle E_n(\lambda)x, y\rangle \tag{20}$$

$$(x, y \in F(A, n)).$$

$$\|A_n x\|^2 = \int_{\mathbb{R}} \lambda^2 d\langle E_n(\lambda)x, y\rangle \tag{21}$$

Due to our preceding lemmas the proof of (20) and (21) now follows standard arguments.

Let be $x \in F(A, n)$, $0 < \varepsilon < 1$ and define $\lambda_i := -n + (i-1)\varepsilon$ for all $0 \leqq i \leqq k$, where the integer k is chosen in such a way that $n \leqq \lambda_k < n + \varepsilon$. With the abbreviations $x_i = E_n(\lambda_i)x - E_n(\lambda_{i-1})x$ for all $1 \leqq i \leqq k$, we have

$$\begin{aligned} &x_1 \in F(A_n + n, \lambda_1 + n) = N(A+n) \\ &x_i \in F(A_n + n, \lambda_i + n) \cap F(A_n + n, \lambda_{i-1} + n)^{\perp} \quad (2 \leqq i \leqq k) \end{aligned} \tag{22}$$

and $x = \sum_{i=1}^{k} x_i$.

Our corollary yields (the case $i = 1$ is trivial)

$$(\lambda_{i-1} + n) \cdot \|x_i\|^2 \leqq \langle (A_n + n)x_i, x_i\rangle \leqq (\lambda_i + n) \cdot \|x_i\|^2$$

resp.

$$(\lambda_{i-1} + n) \cdot \|x_i\| \leqq \|(A_n + n)x_i\| \leqq (\lambda_i + n) \cdot \|x_i\|$$

hence

$$|\langle (A_n - \lambda_i)x_i, x_i\rangle| \leqq \varepsilon \cdot \|x_i\|^2$$

resp.

$$|\langle (A_n+n)^2 x_i - (\lambda_i+n)^2 x_i, x_i\rangle| \leqq \varepsilon(4n+1)\cdot \|x_i\|^2 .$$

Because of (22) we get

$$\left|\langle A_n x, x\rangle - \sum_{i=1}^{k} \lambda_i \langle x_i, x_i\rangle\right| = \left|\sum_{i=1}^{k} \langle (A_n-\lambda_i)x_i, x_i\rangle\right|$$
$$\leqq \varepsilon\cdot \sum_{i=1}^{k} \|x_i\|^2 = \varepsilon\cdot \|x\|^2$$

resp.

$$\left|\langle (A_n+n)^2 x, x\rangle - \sum_{i=1}^{k} (\lambda_i+n)^2 \langle x_i, x_i\rangle\right| \leqq \varepsilon\cdot(4n+1)\cdot\|x\|^2$$

and finally

$$\langle A_n x, x\rangle = \int_{\mathbb{R}} \lambda d\langle E_n(\lambda)x, x\rangle \ , \tag{23}$$

$$\langle (A_n+n)^2 x, x\rangle = \int_{\mathbb{R}} (\lambda+n)^2 d\langle E_n(\lambda)x, x\rangle \tag{24}$$

as $\varepsilon\to 0$. Now (23) and (24) imply (21) as well as (20), the latter by polarization.

The formula (20), which was just proven, tells us that for $l \geqq n$ the family $(E_l(\lambda)|F(A,n))_{\lambda\in\mathbb{R}}$ is the spectral family of A_n, if we are able to prove $E_l(\lambda)(F(A,n)) \subset F(A,n)$, but this follows from the estimates

$$\|A^r E_l(\lambda)x\| = \|(A_l)^r E_l(\lambda)x\| = \|E_l(\lambda)(A_l)^r x\|$$
$$\leqq \|A^r x\| \leqq n^r\|x\| \quad \text{for} \quad x\in F(A,n),\ l\geqq n .$$

Hence (due to uniqueness) we get for any $l\geqq n$

$$E_l(\lambda)P_n = E_n(\lambda)P_n = P_n E_n(\lambda) = E_n(\lambda) \tag{25}$$

and consequently

$$\|E_l(\lambda)x - E_n(\lambda)x\| = \|E_l(\lambda)x - E_l(\lambda)P_n x\| \leqq \|x - P_n x\| \ . \tag{26}$$

Clearly $E(\lambda) := s\text{-}\lim_{n\to\infty} E_n(\lambda)$ exists uniformly with respect to $\lambda\in\mathbb{R}$, so [due to (17), (19)] $(E(\lambda))_{\lambda\in\mathbb{R}}$ is a spectral family.

As $l\to\infty$ we derive from (25)

$$E(\lambda)P_n = E_n(\lambda) \tag{27}$$

and using (15) with A replaced by A_l we get

$$R(E_l[-n,n]) = F(A_l, n) = F(A,n) \quad \text{if} \quad l\geqq n .$$

Thus

$$E[-n,n] = s\text{-}\lim_{l\to\infty} E_l[-n,n] = P_n \ . \tag{28}$$

Picking $x, y \in H$ and using (17), (20), (21), (27), (28) we get

$$\begin{aligned}\langle AP_n x, y\rangle &= \langle AP_n x, P_n y\rangle = \int_{\mathbb{R}} \lambda d\langle E_n(\lambda)P_n x, P_n y\rangle \\ &= \int_{\mathbb{R}} \lambda d\langle E_n(\lambda)P_n x, y\rangle = \int_{\mathbb{R}} \lambda d\langle E(\lambda)P_n x, y\rangle \\ &= \int_{[-n,n]} \lambda d\langle E(\lambda)x, y\rangle\end{aligned}$$

and similarily

$$\|AP_n x\|^2 = \int_{[-n,n]} \lambda^2 d\langle E(\lambda)x, x\rangle .$$

Let us now use Lemma 5 to state

$$x \in D(A) \Leftrightarrow (\|AP_n x\|^2) \text{ converges} \Leftrightarrow \int_{\mathbb{R}} \lambda^2 d\langle E(\lambda)x, x\rangle < \infty .$$

Together with $(x \in D(A), y \in H)$

$$\langle Ax, y\rangle = \lim_{n\to\infty} \langle AP_n x, y\rangle = \lim_{n\to\infty} \int_{[-n,n]} \lambda d\langle E(\lambda)x, y\rangle = \int_{\mathbb{R}} \lambda d\langle E(\lambda)x, y\rangle$$

we complete our proof of the spectral theorem. □

Remarks. First, it should be noted that the proof of Theorem I simplifies considerably, if A is semibounded. For this it suffices to consider $A \geqq 0$. Then we can define the spectral family $(E(\lambda))_{\lambda\in\mathbb{R}}$ of A immediately by

$$E(\lambda) := \begin{cases} \mathrm{Proj}_{F(A,\lambda)} & \text{if } \lambda \geqq 0 \\ 0 & \text{if } \lambda < 0 \end{cases}$$

as in the proof of Theorem I [formula (20)–(24)] one states (somewhat simpler)

$$\left.\begin{aligned}\langle A_n x, y\rangle &= \int_{\mathbb{R}} \lambda \langle dE(\lambda)x, y\rangle \\ \|A_n x\|^2 &= \int_{\mathbb{R}} \lambda^2 d\langle E(\lambda)x, x\rangle\end{aligned}\right\} \quad (x, y \in F(A, n)) .$$

Applying Lemma 5, the spectral theorem is proved. Second, the preceding remarks show that, leaving out of consideration Lemma 4, the case "A semibounded" is as simple (or difficult if you want) as the case "A bounded".

An analysis of the proof of Theorem I shows, that each closed symmetric operator admits (in a suitable sense) a "spectral representation".

Theorem II. *Suppose A is a closed symmetric operator in H. Then there exists one and only one increasing rigth-continuous family $(E(\lambda))_{\lambda\in\mathbb{R}}$ of projections with $E(-\infty)=0$ such that*

i) $\overline{\bigcup_{n=1}^{\infty} F(A, n)} = R(E(\infty))$ *reduces A.*

ii) $A|D(A)\cap R(E(\infty))$ *is the maximal self-adjoint part of A.*

iii) $A|D(A)\cap R(E(\infty)) = \int_{\mathbb{R}} \lambda dE(\lambda)$.

Proof. We only sketch the proof. Use the same notations as in the proof of Theorem I, then $(E(\lambda))_{\lambda\in\mathbb{R}}$ exists. Pick $x\in D(A)$, then $E(\infty)x=s\text{-}\lim_{n\to\infty} P_n x$, $\|AP_n x\| \leqq \|Ax\|$. Thus a subsequence of $(AP_n x)$ converges weakly, so $E(\infty)x\in D(A)$ and $AE(\infty)x\in \overline{\bigcup_{n=1}^{\infty} F(A,n)}=R(E(\infty))$.

Due to [4] p. 278 $R(E(\infty))$ reduces A.

The proof of Theorem I (existence) shows exactly

$$A|D(A)\cap R(E(\infty))=\int_{\mathbb{R}} \lambda dE(\lambda) \qquad [\text{in } R(E(\infty))] ,$$

so $A|D(A)\cap R(E(\infty))$ is a self-adjoint part of A. Suppose A' is a self-adjoint part of A being self-adjoint in the subspace $H'\subset H$, then due to Lemma 4

$$H'=\overline{\bigcup_{n=1}^{\infty} F(A',n)}\subset \overline{\bigcup_{n=1}^{\infty} F(A,n)}=R(E(\infty)),$$

so $A'=A|D(A)\cap H'\subset A|D(A)\cap R(E(\infty))$. Because of iii) the uniqueness of $(E(\lambda))_{\lambda\in\mathbb{R}}$ is clear. □

Concluding Remarks. We wish to notice that the spaces $F(A,\lambda)$ considered here have further applications.

1) Consider for instance self-adjoint operators A_n in the Hilbert space H_n $(1\leqq n\leqq N)$ and a real polynomial P in $\mathbb{R}^N$.

Then the operator $P(A_1,\dots,A_N)$ makes sense on $\bigotimes_{n=1}^{N} D(A_n)$ and a short calculation shows

$$\bigotimes_{n=1}^{N} F(A_n,\lambda)\subset F(\overline{P(A_1,\dots,A_N)}, \tilde{P}(\lambda,\dots,\lambda)) , \tag{29}$$

where $\lambda\geqq 0$, $\overline{P(A_1,\dots,A_N)}$ denotes the closure of $P(A_1,\dots,A_N)$ and $\tilde{P}$ is the same polynomial as P except that each coefficient has been replaced by its absolute value. It follows from (29) and Lemma 4 that $P(A_1,\dots,A_N)$ is essentially self-adjoint on $\bigotimes_{n=1}^{N} D(A_n)$ (compare [10], pp. 247–248).

2) Suppose $(E(\lambda))_{\lambda\in\mathbb{C}}$ is the spectral family of a normal (not necessarily bounded) operator A. Then the uniqueness proof of Theorem I shows that

$$R(E(\{\mu\in\mathbb{C}\,|\,|\mu-\lambda|\leqq\varepsilon\})=F(A-\lambda,\varepsilon)$$

for each $\lambda\in\mathbb{C}$ and each $\varepsilon>0$. Due to Lemma 1 (remark) every bounded operator B satisfying $BA\subset AB$ maps $F(A-\lambda,\varepsilon)$ into itself. Using suitable approximations (compare e.g. [2], p. 68) we get

$$E(\lambda)\cdot B=B\cdot E(\lambda) \qquad (\lambda\in\mathbb{C}) .$$

This is, for bounded operators A, a famous result of Fuglede (see [9], Theorem 1.16).

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Received August 3, 1978, in revised form February 1, 1979