This note describes a unifying principle that can be used to deduce several fundamental theorems in real analysis. It involves the concept of a subordering of the real numbers and states that a subordering that is locally valid with respect to each point of an interval \( I \) coincides with the usual order on \( I \). A number of applications are given followed by an extended version of the theorem.

First we define the concept of a subordering and introduce some notation. Let \( I \) be an interval in \( \mathbb{R} \). A transitive relation \( \triangleleft \) that preserves the natural order \(<\) will be called a *subordering* on \( I \), i.e., \( \triangleleft \) is a subset of \( I \times I \) satisfying:

\[
\begin{align*}
(A1) & \quad \text{If } x \triangleleft y \text{ and } y \triangleleft z \text{ then } x \triangleleft z. \\
(A2) & \quad \text{If } x \triangleleft y \text{ then } x < y.
\end{align*}
\]

Here we use the standard notation \( x \triangleleft y \) if \((x, y) \in \triangleleft\) and we write \( x \not\triangleleft y \) if \((x, y) \notin \triangleleft\). We call a subordering \( \triangleleft \) on the interval \( I \) *locally valid* (with respect to each point \( c \in I \)) if:

\[
(A3) \quad \text{For every } c \in \triangleleft \text{ there is a deleted neighborhood } \hat{V}(c) \text{ of } c \text{ such that } x \triangleleft c \text{ or } c \triangleleft x \text{ whenever } x \in \hat{V}(c).
\]

As usual, if \( V(c) \) is any neighborhood of \( c \) in \( I \) (say \( V(c) = J \cap I \) for some open interval \( J \) containing \( c \)) the deleted neighborhood \( \hat{V}(c) \) is defined to be \( V(c) \setminus \{c\} \).

At this point we make two conventions: First, every relation \( \triangleleft \) appearing in this note is always assumed to preserve the natural ordering, i.e., \( \triangleleft \) is assumed tacitly to be a subset of \( < \) without mentioning this explicitly. Second, all functions in this note are defined on intervals.

**Example 1** Let \( f : I \rightarrow \mathbb{R} \) be a differentiable function satisfying \( f' > 0 \). Take the subordering \( \triangleleft \) on \( I \) to be

\[
x \triangleleft y \iff f(x) < f(y)
\]
Clearly, $\triangleleft$ is locally valid. In fact, we have $\frac{f(x)-f(c)}{x-c} > 0$ for all $x$ in some deleted neighborhood $\hat{V}(c)$, because $f'(c) > 0$.

**Example 2** Let $(\Omega_\alpha)_{\alpha \in A}$ be any open covering of the compact interval $I$ and take the subordering $\triangleleft$ to be

$$x \triangleleft y \iff [x, y] \text{ has the finite covering property with respect to } (\Omega_\alpha)_{\alpha \in A}$$

Since each point $c$ in $I$ has an neighborhood $V(c)$, which is contained in some $\Omega_\alpha$, the subordering $\triangleleft$ is actually locally valid.

**Example 3** Let $f : I \to \mathbb{R}$ be continuous and never zero on $I$. Then, it is clear that

$$x \triangleleft y \iff \text{sgn}(f(x)) = \text{sgn}(f(y))$$

generates a locally valid subordering.

From these examples one is tempted to believe that many different, locally valid suborderings exist. But appearances are deceptive.

**Theorem 1** The natural order $<$ is the only subordering that is locally valid on the interval $I$.

The proof that follows is based on the nested interval axiom. It is just as easy to base a proof of Theorem 1 on the least upper bound property (see Theorem 2).

**Proof.** Let $\triangleleft$ be any locally valid subordering on $\triangleleft$ and suppose there exist points $x$ and $y$ in $I$ satisfying $x < y$ as well as $x \not\triangleleft y$. We consider the point $\frac{x+y}{2}$ and observe that not both relations

$$x < \frac{x+y}{2} \text{ and } \frac{x+y}{2} < y$$

can hold. Hence, we can find points $x_1$ and $y_1$ in $I$ obeying

$$x_1 \not\triangleleft y_1, \quad [x_1, y_1] \subset [x, y] \quad \text{and} \quad y_1 - x_1 = \frac{y - x}{2}.$$  

Continuing this process, we obtain a sequence $([x_n, y_n])_{n \in \mathbb{N}}$ of nested intervals with the property

$$x_n \not\triangleleft y_n, \quad x_n < y_n, \quad y_n - x_n = \frac{y - x}{2^n}.$$  

Let $c$ be the unique point that belongs to all $[x_n, y_n]$. Since $\triangleleft$ is locally valid with respect to $c$,

$$x \triangleleft c \text{ or } c \triangleleft x$$

holds true for all $x$ in some deleted neighborhood $\hat{V}(c)$ of $c$. Moreover, we have

$$[x_n, y_n] \subset V(c)$$
for sufficiently large $n$. If $c = x_n$ or $c = y_n$, we conclude $x_n \prec y_n$ using (2) and (3). Otherwise $x_n < c < y_n$, but then again by (2) and (3) it follows that $x_n \prec c$ as well as $c \prec y_n$. So in any case we deduce $x_n \prec y_n$, contradicting (1). Thus we have proved Theorem 1. □

Taking Theorem 1 into account, we now review the three examples given above. First of all, we state that we have proved in Example 1 the monotonicity of $f$, in Example 2 the Heine-Borel theorem, and in Example 3 Bolzano’s theorem. Second, we notice that each of these classical results has been derived from a single principle. Moreover, our principle itself is a simple consequence of the nested interval axiom (or equivalently of the least upper bound property).

The following further examples illustrate that proofs based on Theorem 1 are often simpler than proofs based on other principles.

**Example 4 (Boundedness of regulated functions)**

Let $f : [a, b] \rightarrow \mathbb{R}$ be a regulated function, i.e. a function having one-sided limits $f_+(c)$ and $f_-(c)$ at each point $c$ in $[a, b]$ (where it makes sense). Consider the locally valid subordering $\prec$ on $[a, b]$ given by

$$x \prec y \iff f([x, y]) \text{ is bounded.}$$

Our principle yields $a \prec b$. Hence, every regulated function is bounded on compact intervals.

**Example 5 (Mean value inequality)**

Let $f : I \rightarrow \mathbb{R}$ be a differentiable function satisfying $m < f' < M$ for some constants $m, M$. Consider the subordering $\prec$ on $I$ generated by

$$x \prec y \iff m \frac{f(y) - f(x)}{y - x} < M \iff m(y - x) < f(y) - f(x) < M(y - x)$$

Since $m < f'(c) < M$, we conclude that $m < \frac{f(x) - f(c)}{x - c} < M$ for all $x$ in some deleted neighborhood $V(c)$ of $c$. Applying Theorem 1 we see that $m < \frac{f(y) - f(x)}{y - x} < M$ if $x < y$ and $x, y \in I$. Hence we have proved the mean value inequality.

**Example 6 (Maximum value theorem)**

Let $I = [a, b]$ be a compact interval and let $f$ be a continuous (or upper continuous) function on $I$ and assume that $f$ does not attain its maximum. Then, for each $c$ in $I$ a point $c'$ in $I$ exists satisfying $f(c) < f(c')$. By continuity of $f$ there is a neighborhood $V(c)$ of $c$ such that $f < f(c')$ on $V(c)$. Thus, we are motivated to introduce a locally valid subordering $\prec$ on $I$ given by

$$x \prec y \iff \text{there is a } d \in f(I) \text{ satisfying } f < d \text{ on } [x, y].$$

Theorem 1 yields $a \prec b$, which is impossible. It follows that a continuous (or...
upper continuous) function on a compact interval has a maximum value.

Example 7 (Uniform continuity of continuous functions)
Let \( f \) be a continuous function from an interval \( I \) into \( \mathbb{R} \). For any positive number \( \epsilon \) we consider the relation \( x \triangleleft y \) defined by

\[
x \triangleleft y \iff f \text{ is uniformly } \epsilon\text{-continuous on } [x, y].
\]

Clearly, \( \triangleleft \) is a locally valid relation. Bearing this in mind, it is easy to see that \( \triangleleft \) is transitive, i.e., a locally valid subordering. Hence, a continuous function is uniformly continuous on every compact interval.

Example 8 (Piecewise linear approximation of continuous functions)
Let \( f \) be a continuous function on an compact interval \( I \). For any fixed positive integer \( n \) we consider the subordering \( x \triangleleft y \) on \( I \) given by

\[
x \triangleleft y \iff \text{there is a piecewise linear function } \varphi : [x, y] \to \mathbb{R} \text{ s.t. } \varphi(x) = f(x), \varphi(y) = f(y) \text{ and } |\varphi - f| \leq 1/n \text{ on } [x, y].
\]

Since \( f \) is continuous at \( c \), we have \( |\varphi - f(c)| \leq 1/2n \) on some neighborhood \( V(c) \) of \( c \). Let \( x \in V(c) \) and assume \( x < c \) (the case \( x > c \) is similar). Consider the linear function \( \varphi : [x, c] \to \mathbb{R} \) determined by the conditions \( \varphi(x) = f(x), \varphi(c) = f(c) \).

Then, \( |\varphi - f| \leq 1/n \) on \( [x, c] \) and thus \( x \triangleleft c \). This shows that \( \triangleleft \) is a locally subordering. Applying Theorem 1 we see that a continuous function on a compact interval can approximated arbitrary closely by piecewise linear functions.

Example 9 (Approximation of regulated functions by step functions)
Similarly let \( f \) be a regulated function defined on a compact interval \( I \). We will show that \( f \) can be approximated uniformly from above (and hence also from below) by a decreasing (increasing) sequence of step functions. Following standard arguments and using the boundedness of \( f(I) \) as a starting point it suffices to show: Given a positive integer \( n \) and a step function \( \alpha : I \to \mathbb{R} \) satisfying \( f \leq \alpha \) there is a step function \( \varphi : I \to \mathbb{R} \) such that \( f \leq \varphi \leq \alpha \) and \( |f - \varphi| \leq 1/n \).

Hence we introduce the subordering \( x \triangleleft y \) on \( I \)

\[
x \triangleleft y \iff \text{there is a step function } \varphi : [x, y] \to \mathbb{R} \text{ which satisfies } f \leq \varphi \leq \alpha \text{ and } |f - \varphi| \leq 1/n \text{ on } [x, y].
\]

Using the fact that \( f \) has one-sided limits it is not hard to see that \( \triangleleft \) is locally valid. Thus, a regulated function can be approximated uniformly from above (below) by a decreasing (increasing) sequence of step functions.

Example 10 (Integrability of continuous (regulated) functions)
Finally, let \( f \) be a continuous real-valued function. Denote by \( J^+(f; x, y) \) the
upper and $J^-(f; x, y)$ the lower integral of $f$ over $[x, y]$ divided by $y - x$. For any positive $\epsilon$

$$x \triangleleft y \iff -\epsilon \leq J^+(f; x, y) - J^-(f; x, y) \leq \epsilon$$

defines a locally valid subordering $\triangleleft = \triangleleft_\epsilon$. Hence, a continuous function is integrable over every compact interval. It should be mentioned that the same arguments above can be applied to regulated functions. Thus regulated functions turn out to be integrable, too.

**Example 11 (Bolzano - Weierstrass theorem)**

Any subset $S$ of $[a, b]$ with no accumulation points is finite.

To see this take the locally valid subordering $\triangleleft$ defined by

$$x \triangleleft y \iff [x, y] \cap S \text{ is finite}$$

**Example 12 (Cantor’s intersection theorem)**

If $(F_n)_{n \in \mathbb{N}}$ is a decreasing sequence of closed subsets of $[a, b]$ whose intersection is empty, then one of the sets $F_n$ must be empty.

Take the locally valid subordering $\triangleleft$ given by

$$x \triangleleft y \iff \exists n \in \mathbb{N} \text{ such that } [x, y] \cap F_n = \emptyset$$

and the result is clear.

In our opinion the suborderings described above are probably the most interesting ones to which Theorem 1 applies. Of course, there are other more or less interesting examples.

**Example 13 (Cousin’s theorem)**

Let $I = [a, b]$ and $\delta : I \to \mathbb{R}^+$ be any gauge on $I$. Then there exists a tagged partition $P$ of $I$ that is $\delta$-fine.

**Proof:** Define the locally valid subordering $\triangleleft$ by

$$x \triangleleft y \iff \exists \text{ a tagged partition of } [x, y] \text{ that is } \delta|[x, y]-\text{fine.}$$

**Example 14 (Darboux property of $f'$)**

Let $f : I \to \mathbb{R}$ be differentiable and $f'$ be never zero on $I$. Then $f' > 0$ or $f' < 0$.

**Proof:** Define the subordering $\triangleleft$ by

$$x \triangleleft y \iff \text{sgn}(f'(x)) = \text{sgn}(f'(y))$$

which is actually locally valid, since $\text{sgn}(f'(x)) \neq \text{sgn}(f'(c))$ would imply that $f$ attains an extremum between $x$ and $c$, which is impossible. Thus $x \triangleleft c$ or $c \triangleleft x$ for $x \in \check{V}(c)$. 5
Example 15 (Potentials along curves)

Given any curl-free vector field $\vec{E} : G \to \mathbb{R}$ and a non-closed, nonintersecting curve $\gamma : [a, b] \to G$. Then there is a potential $\varphi : D \to \mathbb{R}$ in a domain $D \subset G$ containing $\text{supp} \varphi$.

Proof: Take the locally valid subordering $\prec$ defined by

$$x \prec y \iff \exists \text{ a potential of } \vec{E} \text{ in a domain containing } \gamma|[x,y].$$

As a matter of fact, Theorem 1 can be reformulated in an equivalent way which might be more convenient to some readers.

Theorem 1’ A ‘property $\mathcal{P}$’ concerning compact subintervals of a given interval $I$ is valid for all compact intervals of $I$ if

1. ‘property $\mathcal{P}$’ is hereditary, i.e., if both $[x, y]$ and $[y, z]$ have ‘property $\mathcal{P}$’, then $[x, z]$ has ‘property $\mathcal{P}$’.
2. ‘property $\mathcal{P}$’ is locally valid.

As a matter of fact, Theorem 1 shows that a ‘property $\mathcal{P}$’ concerning compact subintervals of a given interval is valid for all compact intervals if

Looking at all those examples presented above, we believe that it is worthwhile to figure out what is the essence of Theorem 1. To do so, let us consider a chain $(X, <)$, i.e., a set $X$ linearly ordered by an antisymmetric relation $<$ on $X$ (see [2, p. 15 for details]). We endow $X$ with the order-topology which has a subbase consisting of all sets of the form $\{x | x < a\}$ or $\{x | a < x\}$ for some $a$ in $X$. As in the case $X = \mathbb{R}$ we call a relation $\preceq$ on $X$ a locally valid subordering if $\preceq$ satisfies (A1)-(A3).

Theorem 2 A chain $(X, <)$ endowed with the order-topology is connected iff $<$ is the only subordering $\preceq$ that is locally valid on $X$.

Proof. Let $x, y \in X, x < y$ and assume that $X$ is connected, i.e., $X$ is order-complete and has no gaps (see [2; pp. 57-58] for details). Define the non-empty set $A = \{z | x \prec z \text{ and } x \leq z \leq y\}$ and consider $\sup A$. In view of (A1) and (A3) we have $\sup A = y$. Using (A3) again, we finally get $x \prec y$. On the other hand, if $X$ is not connected, then $X$ can be written as a disjoint union of two non-empty sets $A_1, A_2$. Define the locally valid subordering $\preceq$ by setting $\preceq = \{(x, y) | x < y; x \text{ and } y \text{ belong to the same } A_i\}$. Then $\preceq$ is strictly contained in $<$. $\square$

Since most theorems in real analysis rederived in this note actually do not involve orderings, it might be interesting to have a substitute for Theorem 1 and 2 where only topological properties enter. This can be done very easily.

Let $X$ be a topological space. We call an equivalence relation $\sim$ locally valid if
(A3) is satisfied. Notice that ∼ is locally valid iff ∼ is continuous in the sense that \( x' \sim y' \), whenever \( x \sim y \) and \( x' \in V(x), y' \in V(y) \) for some neighborhoods \( V(x) \) and \( V(y) \). An equivalence relation is called trivial if every two elements are equivalent.

**Theorem 3** A topological space \( X \) is connected iff every continuous equivalence relation is trivial.

Proof. If ∼ is any continuous equivalence relation on \( X \), then \( X \) is the disjoint union of its open equivalence classes \( [y \mid x \sim y] \). Hence, \( X \) is connected iff ∼ is trivial. □

The broad applicability of our principle presented above leads us to suppose that it is known in the mathematical literature. But we were unable to find more than one reference [4]. Fortunately, during a stay at California Institute of Technology, Professor T. M. Apostol drew my attention to the article [3]. Actually, the article [3] and our note are closely related, but there is also a major difference. The principle presented in [3] is based on an extended Bolzano-Weierstrass theorem, whereas the principle described in this note depends critically on the topological concept of connectedness.

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**References**


